

ONE-PARAMETER FAMILY OF SECOND-ORDER ORDINARY DIFFERENTIAL EQUATIONS

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ABSTRACT. We study one-parameter family of second-order ordinary differential equations.

0. MAIN RESULTS

In this paper, we present 1-parameter family of second-order autonomous ordinary differential equation explicitly given by

$$(1) \quad \frac{d^2 q}{dt^2} = \frac{5}{2q} \left(\frac{dq}{dt} + \eta_2 \right) \left(\frac{dq}{dt} - \eta_2 \right) + \frac{q^2}{2} \left(3\alpha^2 q^5 + 4\alpha\eta_1 q^4 + \eta_1^2 q^3 - 2\alpha\eta_2 q - 4\eta_1\eta_2 \right).$$

Theorem 0.1. *This equation is equivalent to the Hamiltonian system*

$$(2) \quad \frac{dq}{dt} = \frac{\partial H}{\partial p}, \quad \frac{dp}{dt} = -\frac{\partial H}{\partial q}$$

with polynomial Hamiltonian

$$(3) \quad H = (q^5 p + \alpha q^4 + \eta_1 q^3 + \eta_2) p \quad (\eta_1, \eta_2 \in \mathbb{C} - \{0\}).$$

Here q and p denote unknown complex variables and α is a complex parameter.

Proposition 0.1. *This system has (3) as its first integral.*

Theorem 0.2. *The system (2) is invariant under the following birational and symplectic transformation:*

$$(4) \quad s : (q, p; \eta_1, \eta_2, \alpha) \rightarrow (q, p + \frac{\alpha}{q} + \frac{\eta_1}{q^2} + \frac{\eta_2}{q^5}; -\eta_1, -\eta_2, -\alpha).$$

We also present 3-parameter family of second-order ordinary differential equation explicitly given by

$$(5) \quad \begin{aligned} \frac{d^2 q}{dt^2} = & \frac{5}{2q} \left(\frac{dq}{dt} + 1 \right) \left(\frac{dq}{dt} - 1 \right) + \frac{3}{2} (\alpha_1^2 + 2\alpha_1 - 4\alpha_3 + 1) q^7 + 2(\alpha_1 - 2\alpha_2 + 1) t q^6 \\ & + \frac{t^2}{2} q^5 - \alpha_1 q^3 - 2t q^2. \end{aligned}$$

Theorem 0.3. *This equation is equivalent to the Hamiltonian system*

$$(6) \quad \frac{dq}{dt} = \frac{\partial H}{\partial p}, \quad \frac{dp}{dt} = -\frac{\partial H}{\partial q}$$

with polynomial Hamiltonian

$$(7) \quad H = (q^5 p + (\alpha_1 + 1) q^4 + t q^3 + 1) p + \alpha_3 q^3 + \alpha_2 t q^2.$$

We remark that for this system we tried to seek its first integrals of polynomial type with respect to q, p . However, we can not find. Of course, the Hamiltonian (7) is not its first integral.

Theorem 0.4. *The system (7) is invariant under the following birational and symplectic transformation:*

(8)

$$s : (q, p, t; \alpha_1, \alpha_2, \alpha_3) \rightarrow (-(-1)^{\frac{1}{4}}q, -\frac{1}{(-1)^{\frac{1}{4}}}(p + \frac{\alpha_1}{q} + \frac{t}{q^2} + \frac{1}{q^5}), -(-1)^{\frac{3}{4}}t; \\ -\alpha_1, 1 - \alpha_2, \alpha_3 - \alpha_1).$$

We remark that the transformation s satisfies $s^8 = 1$.

1. A GENERALIZATION OF THE SYSTEM (1)

In this section, we also present a generalization of the system (1) explicitly given by

(9)

$$\frac{d^2q}{dt^2} = \frac{n}{2q} \left(\frac{dq}{dt} + \eta_{n-1} \right) \left(\frac{dq}{dt} - \eta_{n-1} \right) \\ - \frac{n}{2} (\alpha x^{n-2} + \eta_1 x^{n-3} + \cdots + \eta_{n-2}) \{x(\alpha x^{n-2} + \eta_1 x^{n-3} + \cdots + \eta_{n-2}) + 2\eta_{n-1}\} \\ + (\alpha x^{n-1} + \eta_1 x^{n-2} + \cdots + \eta_{n-1}) \{ \alpha(n-1)x^{n-2} + \eta_1(n-2)x^{n-3} + \cdots + \eta_{n-2} \} \\ (\eta_1, \eta_2, \dots, \eta_{n-1} \in \mathbb{C} - \{0\}, \quad \alpha \in \mathbb{C}).$$

Theorem 1.1. *This equation is equivalent to the Hamiltonian system (2) with polynomial Hamiltonian*

$$(10) \quad H = (q^n p + \alpha q^{n-1} + \eta_1 q^{n-2} + \cdots + \eta_{n-1})p.$$

Here q and p denote unknown complex variables.

It is easy to see that we obtain

$$(11) \quad p = \frac{1}{2q^n} \left(\frac{dq}{dt} - (\alpha q^{n-1} + \eta_1 q^{n-2} + \cdots + \eta_{n-1}) \right).$$

Proposition 1.1. *This system has (10) as its first integral.*

Proof 1.1.

$$(12) \quad \begin{aligned} \frac{dH}{dt} &= (nq^{n-1}p + \alpha(n-1)q^{n-2} + \cdots + \eta_{n-2})p \frac{dq}{dt} \\ &\quad + (2q^n p + \alpha q^{n-1} + \eta_1 q^{n-2} + \cdots + \eta_{n-1}) \frac{dp}{dt} \\ &= -\frac{dp}{dt} \frac{dq}{dt} + \frac{dq}{dt} \frac{dp}{dt} \\ &= 0. \end{aligned}$$

Theorem 1.2. *The system (10) is invariant under the birational and symplectic transformation:*

(13)

$$s : (q, p; \eta_1, \dots, \eta_{n-1}, \alpha) \rightarrow (q, p + \frac{\alpha}{q} + \frac{\eta_1}{q^2} + \frac{\eta_2}{q^3} + \cdots + \frac{\eta_{n-1}}{q^n}; -\eta_1, \dots, -\eta_{n-1}, -\alpha).$$

Proof 1.2. *Set*

$$Q := q, \quad P := p + \frac{\alpha}{q} + \frac{\eta_1}{q^2} + \frac{\eta_2}{q^3} + \cdots + \frac{\eta_{n-1}}{q^n}.$$

By resolving in q, p , we obtain

$$S : \quad q = Q, \quad p = P - \frac{\alpha}{Q} - \frac{\eta_1}{Q^2} - \frac{\eta_2}{Q^3} - \cdots - \frac{\eta_{n-1}}{Q^n}.$$

By S , we obtain

(14)

$$\begin{aligned} S(H) &= \left(P - \frac{\alpha}{Q} - \frac{\eta_1}{Q^2} - \frac{\eta_2}{Q^3} - \cdots - \frac{\eta_{n-1}}{Q^n} \right) \\ &\quad \{ Q^n \left(P - \frac{\alpha}{Q} - \frac{\eta_1}{Q^2} - \frac{\eta_2}{Q^3} - \cdots - \frac{\eta_{n-1}}{Q^n} \right) + \alpha Q^{n-1} + \eta_1 Q^{n-2} + \cdots + \eta_{n-1} \} \\ &= \left(P - \frac{\alpha}{Q} - \frac{\eta_1}{Q^2} - \frac{\eta_2}{Q^3} - \cdots - \frac{\eta_{n-1}}{Q^n} \right) \\ &\quad \{ Q^n P - \alpha Q^{n-1} - \eta_1 Q^{n-2} - \cdots - \eta_{n-1} + \alpha Q^{n-1} + \eta_1 Q^{n-2} + \cdots + \eta_{n-1} \} \\ &= Q^n P \left(P - \frac{\alpha}{Q} - \frac{\eta_1}{Q^2} - \frac{\eta_2}{Q^3} - \cdots - \frac{\eta_{n-1}}{Q^n} \right) \\ &= (Q^n P - \alpha Q^{n-1} - \eta_1 Q^{n-2} - \cdots - \eta_{n-1}) P \end{aligned}$$

By changing the sign of η_i, α , we obtain the Hamiltonian (10). □

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